

# Places of algebraic function fields in arbitrary characteristic\*

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21. 8. 2003

## Abstract

We consider the Zariski space of all places of an algebraic function field  $F|K$  of arbitrary characteristic and investigate its structure by means of its patch topology. We show that certain sets of places with nice properties (e.g., prime divisors, places of maximal rank, zero-dimensional discrete places) lie dense in this topology. Further, we give several equivalent characterizations of fields that are large, in the sense of F. Pop's Annals paper *Embedding problems over large fields*. We also study the question whether a field  $K$  is existentially closed in an extension field  $L$  if  $L$  admits a  $K$ -rational place. In the appendix, we prove the fact that the Zariski space with the Zariski topology is quasi-compact and that it is a spectral space.

## 1 Introduction

### 1.1 The Zariski space

In this paper, we consider algebraic function fields  $F|K$  of arbitrary characteristic. For any place  $P$  on  $F$ , the valuation ring of  $P$  will be denoted by  $\mathcal{O}_P$ , and its maximal ideal by  $\mathcal{M}_P$ . By a **place of  $F|K$**  we mean a place  $P$  of  $F$  whose restriction to  $K$  is the identity.

Following [Z-SA], we denote by  $S(F|K)$  the set of all valuations (or places) of  $F$  that are trivial on  $K$ . It is called the **Zariski space** (or **Zariski–Riemann manifold**) of  $F|K$ . As in [Z-SA] we shall make no distinction between equivalent valuations, nor between equivalent places; so we are in fact talking about the set of all valuation rings of  $F$  which contain  $K$  (and therefore,  $S(F|K)$  is indeed a set and not a proper class). Also,

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\*The author would like to thank Florian Pop, Alexander Prestel and Niels Schwartz for very inspiring discussions, and Roland Auer for a thorough reading of an earlier version.

This research was partially supported by a Canadian NSERC grant.

AMS Subject Classification: 12J10, 03C60

$S(F|K)$  can be viewed as the set of all places of  $F|K$ , as every place which is trivial on  $K$  (i.e., an isomorphism on  $K$ ) is equivalent to a place whose restriction to  $K$  is the identity.

Let  $\wp$  be a fixed place on  $K$ . The set of all places of  $F$  which extend  $\wp$  will be denoted by  $S(F|K; \wp)$ . Hence,  $S(F|K) = S(F|K; \text{id}_K)$ . Every set  $S(F|K; \wp)$  carries the **Zariski-topology**, for which the basic open sets are the sets of the form

$$\{P \in S(F|K; \wp) \mid a_1, \dots, a_k \in \mathcal{O}_P\}, \quad (1)$$

where  $k \in \mathbb{N} \cup \{0\}$  and  $a_1, \dots, a_k \in F$ .

With this topology,  $S(F|K; \wp)$  is a spectral space (cf. [H]); in particular, it is quasi-compact. Its associated **patch topology** (or **constructible topology**) is the finer topology whose basic open sets are the sets of the form

$$\{P \in S(F|K; \wp) \mid a_1, \dots, a_k \in \mathcal{O}_P; b_1, \dots, b_\ell \in \mathcal{M}_P\}, \quad (2)$$

where  $k, \ell \in \mathbb{N} \cup \{0\}$  and  $a_1, \dots, a_k, b_1, \dots, b_\ell \in F$ . With the patch topology,  $S(F|K; \wp)$  is a totally disconnected compact Hausdorff space. For the convenience of the reader, we include a proof for the compactness in the appendix. We derive it from a more general model theoretic framework. See [Z-SA] or [V] for the more classical proofs of the quasi-compactness of  $S(F|K)$  under the Zariski topology, and for further details about  $S(F|K)$ . See also [Z1] and [Z2] for further information and the application of  $S(F|K)$  to algebraic geometry.

If  $P$  is any place on a field  $L$ , then we will denote by  $LP$  its residue field, by  $v_P$  an associated valuation (unique up to equivalence) and by  $v_P L$  its value group. For every place  $P \in S(F|K; \wp)$ , its residue field  $FP$  contains  $K\wp$ ; we set  $\dim P := \text{trdeg } FP|K\wp$ . Further, its value group  $v_P F$  contains  $v_\wp K$ . We set  $\text{rr } P := \dim_{\mathbb{Q}} \mathbb{Q} \otimes (v_P F / v_\wp K)$ ; this is called the **rational rank** of  $v_P F / v_\wp K$ . Then we have the well known **Abhyankar inequality**:

$$\text{trdeg } F|K \geq \dim P + \text{rr } P. \quad (3)$$

If equality holds, then we call  $P$  an **Abhyankar place**. The value group  $v_P F$  of every Abhyankar place  $P$  is finitely generated modulo  $v_\wp K$ , and the residue field  $FP$  is a finitely generated extension of  $K\wp$  (see Corollary 26 below).

As soon as the transcendence degree of  $F|K$  is bigger than 1, the Zariski space  $S(F|K)$  will contain “bad places” (cf. [K5]). The residue field  $FP$  is then not necessarily finitely generated over  $K\wp = K$  (even though  $F$  is finitely generated over  $K$ ). This can be a serious obstruction in the search for a  $K$ -rational specialization of  $P$ ; cf. [J-R]. (We call a place  $P$  of a field  $L$   **$K$ -rational** if  $K$  is a subfield of  $L$ ,  $P$  is trivial on  $K$ , and  $LP = K$ .) Similarly, the value group  $v_P F$  of  $P$  may not be finitely generated. For such bad places, local uniformization is much more difficult than for Abhyankar places. In fact, we show in [K-K] that *Abhyankar places  $P$  of  $F|K$  admit local uniformization in arbitrary characteristic, provided that  $FP|K$  is separable*. To the other extreme, we also prove local

uniformization for rational discrete places ([K6]). (We call a place  $P \in S(F|K)$  **discrete** if its value group is isomorphic to  $\mathbb{Z}$ , and **rational** if  $FP = K$ .) For most other places, we have not yet been able to obtain local uniformization without taking finite extensions of the function field into the bargain ([K6]).

Therefore, the question arises whether we can “replace bad places  $Q$  by good places  $P$ ”. Certainly, in doing so we want to keep a certain amount of information unaltered. For instance, we could fix finitely many elements on which  $Q$  is finite and require that also  $P$  is finite on them. This amounts to asking whether the “good” places lie Zariski-dense in the Zariski space. But if we mean by a “good place” just a place with finitely generated value group and residue field finitely generated over  $K$ , then the answer is trivial: the identity is a suitable place, as it lies in every Zariski-open neighborhood. The situation becomes non-trivial when we work with the patch topology instead of the Zariski topology. In addition, we can even try to keep more information on values or residues, e.g., rational (in)dependence of values or algebraic (in)dependence of residues.

In [K–P], such problems are solved in the case of  $\text{char } K = 0$  by an application of the Ax–Kochen–Ershov–Theorem. In the present paper, we will prove similar results for arbitrary characteristic, using the model theory of tame fields, which we introduced in [K1] ([K2], [K7]).

## 1.2 Dense subsets of the Zariski space

When we say “dense” we will always mean “dense with respect to the patch topology”. Throughout this section, we let  $F|K$  be a function field of transcendence degree  $n$ , and  $\wp$  a place on  $K$ . We set  $p = \text{char } K$  if this is positive, and  $p = 1$  otherwise.

Our key result is the following generalization of the Main Theorem of [K–P]. Take any ordered abelian group  $\Gamma$  and  $r \in \mathbb{N}$ . If the direct product  $\Gamma \oplus \bigoplus_r \mathbb{Z}$  of  $\Gamma$  with  $r$  copies of  $\mathbb{Z}$  is equipped with an arbitrary extension of the ordering of  $\Gamma$ , then it will be called an  $r$ -extension of  $\Gamma$ .

An extension  $(K_1, P_1) \subseteq (K_2, P_2)$  is called **immediate** if the canonical embedding of  $K_1 P_1$  in  $K_2 P_2$  and the canonical embedding of  $v_{P_1} K_1$  in  $v_{P_2} K_2$  are onto.

**Theorem 1** *Take a place  $Q \in S(F|K; \wp)$  and*

$$a_1, \dots, a_m \in F .$$

*Then there exists a place  $P \in S(F|K; \wp)$  with value group finitely generated over  $v_\wp K$  and residue field finitely generated over  $K_\wp$ , such that*

$$a_i P = a_i Q \quad \text{and} \quad v_P a_i = v_Q a_i \quad \text{for} \quad 1 \leq i \leq m .$$

*Moreover, if  $r_1$  and  $d_1$  are natural numbers satisfying*

$$\dim Q \leq d_1 , \quad \text{rr } Q \leq r_1 \quad \text{and} \quad 1 \leq r_1 + d_1 \leq n ,$$

then  $P$  may be chosen to satisfy in addition:

(a)  $\dim P = d_1$  and  $FP$  is a subfield of the rational function field in  $d_1 - \dim Q$  variables over the perfect hull of  $FQ$ ,

(b)  $\text{rr } P = r_1$  and  $v_P F$  is a subgroup of an arbitrarily chosen  $(r_1 - \text{rr } Q)$ -extension of the  $p$ -divisible hull of  $v_Q F$ .

The above remains true even for  $d_1 = 0 = r_1$ , provided that each finite extension of  $(K, \wp)$  admits an immediate extension of transcendence degree  $n$ .

The last condition mentioned in the theorem holds for instance for all  $(K, \wp)$  for which the completion is of transcendence degree at least  $n$ . Note that the case of  $d_1 = 0 = r_1$  only appears when  $\wp$  is nontrivial.

If  $v_Q a_i \geq 0$  for  $1 \leq i \leq k$ , and if  $b_1, \dots, b_\ell \in F$  such that  $v_Q b_j > 0$  for  $1 \leq j \leq \ell$ , then we can choose  $P$  according to the theorem such that also  $v_P a_i \geq 0$  for  $1 \leq i \leq k$  and  $v_P b_j > 0$  for  $1 \leq j \leq \ell$ . That is, we can find a  $P$  with the required properties in every open neighborhood of  $Q$  w.r.t. the patch topology.

If we choose  $r_1 = n - d_1$  then  $P$  will be an Abhyankar place. Hence our theorem yields:

**Corollary 2** *The set of all places with finitely generated value group modulo  $v_\wp K$  and with residue field finitely generated over  $K_\wp$  lies dense in  $S(F|K; \wp)$ . The same holds for its subset of all Abhyankar places.*

In certain cases we would like to obtain value groups of smaller rational rank; e.g., we may want to get discrete places in the case where  $\wp$  is trivial. A modification in the proof of Theorem 1 yields the following result:

**Theorem 3** *Take a place  $Q \in S(F|K; \wp)$  and  $a_1, \dots, a_m \in F$ . Choose  $r_1$  and  $d_1$  such that*

$$\dim Q \leq d_1 \leq n - 1 \quad \text{and} \quad 1 \leq r_1 \leq n - d_1 .$$

*Then there is a place  $P$  such that*

$$a_i P = a_i Q \quad \text{for} \quad 1 \leq i \leq m$$

*and*

(a)  $\dim P = d_1$  and  $FP$  is a subfield, finitely generated over  $K_\wp$ , of a purely transcendental extension of transcendence degree  $d_1 - \dim Q$  over the perfect hull of  $FQ$ ,

(b)  $\text{rr } P = r_1$  and  $v_P F$  is a subgroup, finitely generated over  $v_\wp K$ , of an arbitrarily chosen  $r_1$ -extension of the divisible hull of  $v_Q K$ .

The above remains true even for  $r_1 = 0$ , provided that each finite extension of  $(K, \wp)$  admits an immediate extension of transcendence degree  $n$ .

We deduce two corollaries for the Zariski space  $S(F|K)$ ; we leave it to the reader to formulate analogous results for  $S(F|K; \wp)$ . A place  $P \in S(F|K)$  of dimension  $\text{trdeg } F|K - 1$  is called a **prime divisor of  $F|K$**  (one also says that  $P$  has **codimension 1**). Every prime divisor is an Abhyankar place, has value group isomorphic to  $\mathbb{Z}$  and a residue field which is finitely generated over  $K$  (cf. Lemma 25 below). From the above theorem, applied with  $d_1 = n - 1$ , we obtain the following result:

**Corollary 4** *The prime divisors of  $F|K$  lie dense in  $S(F|K)$ .*

If on the other hand we choose  $d_1 = \dim Q$ , then  $FP$  will be contained in a finite purely inseparable extension of  $FQ$ . If  $\dim Q = 0$ , i.e.,  $FQ|K$  is algebraic, then it follows that  $FP$  is a finite extension of  $K$ . If in addition  $K$  is perfect and  $Q$  is rational, then also  $P$  is rational. With  $r_1 = 1$ , we obtain:

**Corollary 5** *If  $K$  is perfect, then the discrete rational places lie dense in the space of all rational places of  $F|K$ .*

A place  $P \in S(F|K; \wp)$  is called **rational** if  $FP = K\wp$ . Further,  $P$  is called a **place of maximal rank** if  $v_\wp K$  is a convex subgroup of  $v_P F$ , which implies that the ordering of  $v_P F$  canonically induces an ordering on  $v_P F/v_\wp K$ , and the rank  $\text{rk } v_P F/v_\wp K$  of  $v_P F/v_\wp K$  with respect to the induced ordering (the number of proper convex subgroups of  $v_P F/v_\wp K$ ) is equal to the transcendence degree of  $F|K$ . Since the rational rank is always bigger than or equal to the rank, inequality (5) of Corollary 26 shows that  $\text{rk } v_P F/v_\wp K \leq \text{trdeg } F|K$ . Every place of maximal rank is a zero-dimensional Abhyankar place. We take  $r_1 = \text{trdeg } F|K$  and  $\bigoplus_{r_1} \mathbb{Z} \oplus v_\wp K$  to be lexicographically ordered (that is,  $v_\wp K$  is a convex subgroup of  $\bigoplus_{r_1} \mathbb{Z} \oplus v_\wp K$  and  $(\bigoplus_{r_1} \mathbb{Z} \oplus v_\wp K)/v_\wp K$  is of rank  $r_1$ ). Then we obtain from Theorem 3:

**Corollary 6** *If  $K\wp$  is perfect, then the rational places of maximal rank lie dense in the subspace of all rational places in  $S(F|K; \wp)$ .*

In order to decrease the dimension of places, we complement Theorem 3 by the following theorem, which we will prove in Section 3.3:

**Theorem 7** *Take a place  $Q \in S(F|K; \wp)$  and  $a_1, \dots, a_m \in F$ . Assume that  $\dim Q > 0$ . Choose  $r_1$  and  $d_1$  such that*

$$\text{rr } Q + 1 \leq r_1 \leq n \quad \text{and} \quad 0 \leq d_1 \leq n - r_1 .$$

*Then there is a place  $P$  such that*

$$v_P a_i = v_Q a_i \quad \text{for } 1 \leq i \leq m \tag{4}$$

*and*

(a)  $\dim P = d_1$  and  $FP$  is a subfield, finitely generated over  $K_\wp$ , of a purely transcendental extension of the algebraic closure of  $FQ$ ,

(b)  $\text{rr } P = r_1$  and  $v_P F$  is a subgroup, finitely generated over  $v_\wp K$ , of an arbitrarily chosen  $(r_1 - \text{rr } Q - 1)$ -extension of a group  $\Gamma$  which admits  $\mathbb{Z}$  as a convex subgroup such that  $\Gamma/\mathbb{Z}$  is isomorphic to a subgroup of the  $p$ -divisible hull of  $v_Q F$  which is also finitely generated over  $v_\wp K$ .

Here, assertion (4) means that there is an embedding  $\iota$  of the group  $\sum_{i=1}^m \mathbb{Z} v_Q a_i$  in  $v_P F$  such that  $v_P a_i = \iota v_Q a_i$ .

Taking  $\wp = \text{id}_K$  and  $d_1 = 0$ , we obtain:

**Corollary 8** *The zero-dimensional places with finitely generated value group and residue field a finite extension of  $K$  lie dense in  $S(F|K)$ .*

Applying first Theorem 7 and then Theorem 3, we obtain:

**Theorem 9** *Take  $d_1, r_1 \in \mathbb{N}$  such that  $d_1 \geq 0$ ,  $r_1 \geq 1$  and  $d_1 + r_1 \leq \text{trdeg } F|K$ . Then the places  $P$  with*

(a) *residue field  $FP$  a subfield of a purely transcendental extension of the algebraic closure of  $K_\wp$ , finitely generated of transcendence degree  $d_1$  over  $K_\wp$ , and*

(b) *value group  $v_P F$  a subgroup of some fixed  $r_1$ -extension of the divisible hull of  $v_\wp K$  such that  $v_P F/v_\wp K$  is of rational rank  $r_1$  and finitely generated, lie dense in  $S(F|K; \wp)$ .*

Taking  $\wp = \text{id}_K$ ,  $d_1 = 0$  and  $r_1 = 1$ , we obtain:

**Corollary 10** *The discrete zero-dimensional places with residue field a finite extension of  $K$  lie dense in  $S(F|K)$ .*

If we apply Theorem 9 with  $d_1 = 0$  and  $r_1 = \text{trdeg } F|K$ , where we take the  $r_1$ -extension  $\bigoplus_{r_1} \mathbb{Z} \oplus v_\wp K$  to be lexicographically ordered, we obtain:

**Corollary 11** *The zero-dimensional places of maximal rank with residue field a finite extension of  $K_\wp$  lie dense in  $S(F|K; \wp)$ .*

Corollary 5 and Corollary 6 state that if  $K$  is perfect, then the discrete rational places and the rational places of maximal rank lie dense in the space of all rational places. We do not know whether Corollary 5 and Corollary 6 hold without the assumption that  $K$  be perfect. We need this assumption to deduce these results from Theorem 3. But if there is a rational place which admits a strong form of local uniformization, then we can show the same result without this assumption. We will say that a place  $P$  of the function field  $F|K$  admits **smooth local uniformization** if there is a model of  $F|K$  on which

$P$  is centered at a smooth point; in addition, we require that if finitely many elements  $a_1, \dots, a_m \in \mathcal{O}_P$  are given, then the model can be chosen in such a way that they are included in the coordinate ring. If  $K$  is perfect, this is equivalent to the usual notion of local uniformization where “simple point” is used instead of “smooth point”. In [K6] and [K–K] we show:

**Theorem 12** *Every rational discrete and every rational Abhyankar place admits smooth local uniformization.*

The following density result will be proved in Section 3.4:

**Theorem 13** *The rational discrete places and the rational places of maximal rank lie dense in the space of all rational places of  $F|K$  which admit smooth local uniformization.*

### 1.3 Large fields

Following F. Pop [POP1,2], a field  $K$  is called a **large field** if it satisfies one of the following equivalent conditions:

(LF) *For every smooth curve over  $K$  the set of rational points is infinite if it is non-empty.*

(LF') *In every smooth, integral variety over  $K$  the set of rational points is Zariski-dense if it is non-empty.*

(LF'') *For every function field  $F|K$  in one variable the set of rational places is infinite if it is non-empty.*

For the equivalence of (LF) and (LF'), note that the set of all smooth  $K$ -curves through a given smooth  $K$ -rational point of an integral  $K$ -variety  $X$  is Zariski-dense in  $X$ . If (LF) holds, then the set of  $K$ -rational points of any such curve is Zariski-dense in the curve, which implies that the set of  $K$ -rational points of  $X$  is Zariski-dense in  $X$ . The equivalence of (LF) and (LF'') follows from two well-known facts: a) every function field in one variable is the function field of a smooth curve (cf. [HA], Chap. I, Theorem 6.9), and b) every  $K$ -rational point of a smooth curve gives rise to a  $K$ -rational place. The latter is a special case of a much more general result which we will need later:

**Theorem 14** *Assume that the affine irreducible variety  $V$  defined over  $K$  has a simple  $K$ -rational point. Then its function field admits a rational place of maximal rank, centered at this point.*

This follows from results in [A] (see appendix A of [J–R]).

A field  $K$  is existentially closed in an extension field  $L$  if for every  $m, n \in \mathbb{N}$  and every choice of polynomials  $f_1, \dots, f_n, g \in K[X_1, \dots, X_m]$ , whenever  $f_1, \dots, f_n$  have a common zero in  $L^m$  which is not a zero of  $g$ , they also have a common zero in  $K^m$  which is not a zero of  $g$ . In Section 3.5, we shall prove:

**Theorem 15** *The following conditions are equivalent:*

- 1)  $K$  is a large field,
- 2)  $K$  is existentially closed in every function field  $F$  in one variable over  $K$  which admits a  $K$ -rational place,
- 3)  $K$  is existentially closed in the henselization  $K(t)^h$  of the rational function field  $K(t)$  with respect to the  $t$ -adic valuation,
- 4)  $K$  is existentially closed in the field  $K((t))$  of formal Laurent series,
- 5)  $K$  is existentially closed in every extension field which admits a discrete  $K$ -rational place.

The canonical  $t$ -adic place of the fields  $K(t)^h$  and  $K((t))$  is discrete, and it is trivial on  $K$  and  $K$ -rational. Therefore, 5) implies 3) and 4).

In [L], Serge Lang proved that every field  $K$  complete under a rank one valuation is large. But this already follows from the fact that such a field is henselian. Indeed, if a field  $K$  admits a non-trivial henselian valuation, then the Implicit Function Theorem holds in  $K$  (cf. [P–Z]). Using this fact, it is easy to show that  $K$  satisfies (LF). On the other hand, it is also easy to prove that  $K$  satisfies condition 3) of the foregoing theorem, and we will give the proof in Section 3.5 to demonstrate the arguments that are typical for this model theoretic approach. We note:

**Proposition 16** *If a field  $K$  admits a non-trivial henselian valuation, then it is large.*

In view of condition 5) of the above theorem, the question arises whether the existence of a  $K$ -rational place of an extension field  $L$  of a large field  $K$  always implies that  $K$  is existentially closed in  $L$ . We will discuss this in the next section.

## 1.4 Rational place = existentially closed?

Condition 4) of Theorem 15 leads us to ask whether large fields satisfy conditions which may appear to be even stronger. In fact, we shall prove in Section 3.5:

**Theorem 17** *Let  $K$  be a perfect field. Then the following conditions are equivalent:*

- 1)  $K$  is a large field,
- 2)  $K$  is existentially closed in every power series field  $K((G))$ .
- 3)  $K$  is existentially closed in every extension field  $L$  which admits a  $K$ -rational place.

In particular, we obtain:

**Theorem 18** *Let  $K$  be a perfect field which admits a henselian valuation. Assume that the extension field  $L$  of  $K$  admits a  $K$ -rational place. Then  $K$  is existentially closed in  $L$ .*

For a general field  $K$ , we do not know whether conditions 1), 2) and 3) of this theorem are equivalent. Note that in general, not every field  $L$  as in 3) is embeddable in a power series field  $K((G))$  (as we are not admitting non-trivial factor systems here; cf. [KA]). Hence it is an interesting question whether 2) and 3) are *always* equivalent.

A field  $K$  is existentially closed in an extension field  $L$  if it is existentially closed in every finitely generated subextension  $F$  in  $L$ . If  $L$  admits a  $K$ -rational place  $P$ , then every such function field  $F$  admits a  $K$ -rational place, namely, the restriction of  $P$ . Hence, condition 3) of the foregoing theorem is equivalent to the following condition on  $K$ :

**(RP=EC)** *If an algebraic function field  $F|K$  admits a rational place, then  $K$  is existentially closed in  $F$ .*

Note that in contrast to condition 2) of Theorem 15, we are not restricting our condition to function fields in one variable here.

By Theorem 15, every field  $K$  which satisfies (RP=EC) is large. Let us see what we can say about the converse. Take a function field  $F|K$  with a rational place  $P$  which admits local uniformization. That is,  $F|K$  admits a model on which  $P$  is centered at a simple  $K$ -rational point. By Theorem 14,  $F$  also admits a  $K$ -rational place  $Q$  of maximal rank. By Theorem 12,  $Q$  admits smooth local uniformization. Hence by Theorem 13,  $F|K$  also admits a rational discrete place. If  $K$  is large, then it follows from Theorem 15 that  $K$  is existentially closed in  $F$ . This proves the following well known result:

**Theorem 19** *Let  $K$  be a large field and  $F|K$  an algebraic function field. If there is a rational place of  $F|K$  which admits local uniformization, then  $K$  is existentially closed in  $F$ .*

As an immediate consequence, we obtain:

**Theorem 20** *Assume that all rational places of arbitrary function fields admit local uniformization. Then every large field satisfies (RP=EC), and the three conditions of Theorem 17 are equivalent, for arbitrary fields  $K$ .*

Theorem 19 together with Theorem 12 implies:

**Corollary 21** *Let  $K$  be a large field and  $F|K$  an algebraic function field. If there is a rational discrete or a rational Abhyankar place of  $F|K$ , then  $K$  is existentially closed in  $F$ .*

For the case of  $F|K$  admitting a rational discrete place  $P$ , the assertion is already contained in Theorem 15.

**Remark 22** In [ER], Yuri Ershov proves the following: *If  $K$  admits a henselian valuation and  $V$  is an algebraic variety over  $K$  with function field  $F$  which admits a rational generalized discrete place  $P$ , then  $V$  has a simple rational point.* It then follows by Theorem 14 and Corollary 21 that  $K$  is existentially closed in  $F$ . Here, “generalized discrete” means

that the value group is the lexicographic product of copies of  $\mathbb{Z}$ . Ershov also observes that if  $\text{char } K = 0$  or a weak form of local uniformization holds in positive characteristic, then one does not need any condition on the value group of the rational place.

To conclude with, let us state the converse of our above results. The following is a generalization of the lemma on p. 190 of [K–P]:

**Theorem 23** *Let  $F|K$  be an algebraic function field such that  $K$  is existentially closed in  $F$ . Take any elements  $z_1, \dots, z_n \in F$ . Then there are infinitely many rational places of  $F|K$  of maximal rank which are finite on  $z_1, \dots, z_n$ .*

Note the analogy between this theorem and Theorem 14.

## 1.5 The key ingredient for the proof of the main theorem

The key ingredient in our proofs of Theorem 1 and Theorem 3 is our generalization of the Ax–Kochen–Ershov Theorem to the class of all tame fields. A **tame field** is a henselian valued field  $(K, v)$  for which the ramification field  $K^r$  of the normal extension  $(K^{\text{sep}}|K, v)$  is algebraically closed. Here,  $K^{\text{sep}}$  denotes the separable algebraic closure. It follows from the definition that for a tame field  $(K, v)$ ,  $K^{\text{sep}}$  is algebraically closed, i.e.,  $K$  is perfect. For further basic properties of tame fields, see Section 2.2.

By  $P_v$  we denote the place associated with  $v$ . The following theorem was proved in [K1] (and will be published in [K2], [K7]):

**Theorem 24** *Every tame field  $(K, v)$  satisfies the following Ax–Kochen–Ershov principle: If  $(K, v) \subset (F, v)$  is an extension of valued fields,  $KP_v$  is existentially closed in  $FP_v$  (in the language of fields), and  $vK$  is existentially closed in  $vF$  (in the language of ordered groups), then  $(K, v)$  is existentially closed in  $(F, v)$  (in the language of valued fields).*

For the meaning of “existentially closed” in the setting of valued fields and of ordered abelian groups, see [K–P].

Note that for the proof of Corollary 2 we would only need Abraham Robinson’s theorem on the model completeness of the theory of algebraically closed valued fields (cf. [RO]). But we need the above theorem in order to obtain assertions (a) and (b) in Theorems 1, 3 and 7.

## 2 Some preliminaries

For basic facts from valuation theory, see [EN], [RI], [W], [Z–SA], [K2].

## 2.1 Valuation independence

For the easy proof of the following theorem, see [BO], Chapter VI, §10.3, Theorem 1, or [K2].

**Lemma 25** *Let  $L|K$  be an extension of fields and  $v$  a valuation on  $L$  with associated place  $P_v$ . Take elements  $x_i, y_j \in L$ ,  $i \in I$ ,  $j \in J$ , such that the values  $vx_i$ ,  $i \in I$ , are rationally independent over  $vK$ , and the residues  $y_j P_v$ ,  $j \in J$ , are algebraically independent over  $KP_v$ . Then the elements  $x_i, y_j$ ,  $i \in I$ ,  $j \in J$ , are algebraically independent over  $K$ . Moreover,*

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) &= vK \oplus \bigoplus_{i \in I} \mathbb{Z}vx_i, \\ K(x_i, y_j \mid i \in I, j \in J)P_v &= KP_v(y_j P_v \mid j \in J). \end{aligned}$$

*The valuation  $v$  on  $K(x_i, y_j \mid i \in I, j \in J)$  is uniquely determined by its restriction to  $K$ , the values  $vx_i$  and the residues  $y_j P_v$ .*

*Conversely, let  $(K, v)$  be any valued field,  $\alpha_i$  values in some ordered abelian group extension of  $vK$ , and  $\xi_j$  elements in some field extension of  $KP_v$ . Then there exists an extension of  $v$  to the purely transcendental extension  $K(x_i, y_j \mid i \in I, j \in J)$  such that  $vx_i = \alpha_i$  and  $y_j P_v = \xi_j$ .*

**Corollary 26** *Let  $L|K$  be an extension of finite transcendence degree, and  $v$  a valuation on  $L$ . Then*

$$\text{trdeg } L|K \geq \text{trdeg } LP_v|KP_v + \dim_{\mathbb{Q}} \mathbb{Q} \otimes (vL/vK). \quad (5)$$

*If in addition  $L|K$  is a function field and if equality holds in (5), then  $vL/vK$  and  $LP_v|KP_v$  are finitely generated.*

*Proof:* Choose elements  $x_1, \dots, x_\rho, y_1, \dots, y_\tau \in L$  such that the values  $vx_1, \dots, vx_\rho$  are rationally independent over  $vK$  and the residues  $y_1 P_v, \dots, y_\tau P_v$  are algebraically independent over  $KP_v$ . Then by the foregoing lemma,  $\rho + \tau \leq \text{trdeg } L|K$ . This proves that  $\text{trdeg } LP_v|KP_v$  and the rational rank of  $vL/vK$  are finite. Therefore, we may choose the elements  $x_i, y_j$  such that  $\tau = \text{trdeg } LP_v|KP_v$  and  $\rho = \dim_{\mathbb{Q}} \mathbb{Q} \otimes (vL/vK)$  to obtain inequality (5).

Assume that this is an equality. This means that for  $L_0 := K(x_1, \dots, x_\rho, y_1, \dots, y_\tau)$ , the extension  $L|L_0$  is algebraic. Since  $L|K$  is finitely generated, it follows that this extension is finite. This yields that  $vL/vL_0$  and  $LP_v|L_0 P_v$  are finite (cf. [EN], [RI] or [BO]). Since already  $vL_0/vK$  and  $L_0 P_v|KP_v$  are finitely generated by the foregoing lemma, it follows that also  $vL/vK$  and  $LP_v|KP_v$  are finitely generated.  $\square$

The proof of the following fact will be published in [K8]:

**Proposition 27** *Let  $L|K$  be an extension of finite transcendence degree, and  $v$  a non-trivial valuation on  $L$ . If  $\text{trdeg } LP_v|KP_v \geq 1$  or  $\dim_{\mathbb{Q}} \mathbb{Q} \otimes (vL/vK) \geq 1$ , then  $(L, v)$  admits an immediate extension of infinite transcendence degree.*

## 2.2 Tame fields

We will now discuss basic properties of tame fields. Let  $(K, v)$  be a valued field. Recall that we set  $p = \text{char } KP_v$  if this is positive, and  $p = 1$  if  $\text{char } K = 0$ . By ramification theory,  $K^{\text{sep}}|K^r$  is a  $p$ -extension. Hence if  $(K, v)$  is a henselian field of residue characteristic  $\text{char } KP_v = 0$ , then this extension is trivial. Since then also  $\text{char } K = 0$ , it follows that  $K^r = K^{\text{sep}} = \tilde{K}$ , the algebraic closure of  $K$ . Therefore,

**Lemma 28** *Every henselian field of residue characteristic 0 is a tame field.*

Suppose that  $K_1|K$  is a subextension of  $K^r|K$ . Then  $K_1^r = K^r$ . This proves:

**Lemma 29** *Every algebraic extension of a tame field is again a tame field.*

A valued field  $(K, v)$  is called **algebraically maximal** if it admits no proper immediate algebraic extension. Since the henselization is an immediate algebraic extension, every algebraically maximal field is henselian. We give a characterization for tame fields (the proof will be published in [K2], [K7]):

**Lemma 30** *The following assertions are equivalent:*

- 1)  $(K, v)$  is a tame field,
- 2)  $(K, v)$  is algebraically maximal,  $vK$  is  $p$ -divisible and  $KP_v$  is perfect.

*If in addition  $\text{char } K = \text{char } KP_v$ , then the above assertions are also equivalent to*

- 3)  $(K, v)$  is algebraically maximal and perfect.

**Corollary 31** *Assume that  $\text{char } K = \text{char } KP_v$ . Then every maximal immediate algebraic extension of the perfect hull of  $(K, v)$  is a tame field.*

Assume that  $\text{char } K = p > 0$ , and let  $K^{1/p^\infty}$  denote the perfect hull of  $K$ . There is a unique extension of every valuation  $v$  from  $K$  to  $K^{1/p^\infty}$ , which we will again denote by  $v$ . The value group  $vK^{1/p^\infty}$  is the  $p$ -divisible hull of  $vK$ , and the residue field  $K^{1/p^\infty}P_v$  is the perfect hull of  $KP_v$ . But even if  $\text{char } K \neq p$ , Section 2.3 of [K5] shows that it is easy to construct an algebraic extension  $(K', v)$  such that

- $vK' = \frac{1}{p^\infty}vK$  (the  $p$ -divisible hull of  $vK$ ), and
- $K'P_v = (KP_v)^{1/p^\infty}$  (the perfect hull of  $KP_v$ ).

If these two assertions hold, then by Lemma 30, every maximal immediate algebraic extension  $(L, v)$  of  $(K', v)$  is a tame field. We have proved:

**Proposition 32** *For every valued field  $(K, v)$ , there is an algebraic extension  $(L, v)$  which is a tame field and satisfies*

$$vL = \frac{1}{p^\infty}vK \quad \text{and} \quad LP_v = (KP_v)^{1/p^\infty}. \quad (6)$$

The following is a crucial lemma in the theory of tame fields. It was proved in [K1] (the proof will be published in [K2], [K7]).

**Lemma 33** *Let  $(L, v)$  be a tame field and  $K \subset L$  a relatively algebraically closed subfield. If in addition  $LP_v|KP_v$  is an algebraic extension, then  $(K, v)$  is also a tame field and moreover,  $vL/vK$  is torsion free and  $KP_v = LP_v$ .*

### 3 Proof of the main theorems

#### 3.1 Proof of Theorem 1

Assume that  $F|K$  is an algebraic function field of transcendence degree  $n$ ,  $\wp$  a place on  $K$ ,  $Q \in S(F|K; \wp)$ , and  $a_1, \dots, a_m \in F$ . We set

$$d := \dim Q \quad \text{and} \quad r := \text{rr } Q.$$

Then we choose  $y_1, \dots, y_d \in F$  such that  $y_1Q, \dots, y_dQ$  form a transcendence basis of  $FQ|K_\wp$ . Further, we choose  $x_1, \dots, x_r \in F$  such that the values  $v_Q x_1, \dots, v_Q x_r$  form a maximal set of rationally independent elements in  $v_Q F$  modulo  $v_\wp K$ . According to Lemma 25, the elements  $x_1, \dots, x_r, y_1, \dots, y_d$  are algebraically independent over  $K$ . We take  $K_0$  to be the rational function field  $K(x_1, \dots, x_r, y_1, \dots, y_d)$ .

By Proposition 32 we choose an algebraic extension  $(L, Q)$  of  $(F, Q)$  which is a tame field and satisfies (6), for  $F$  in the place of  $K$ . By construction of  $K_0$ , we have that  $v_Q L/v_Q K_0$  is a torsion group and  $LQ|K_0Q$  is algebraic.

Let  $K'$  be the relative algebraic closure of  $K_0$  in  $L$ , and let  $Q'$  be the restriction of  $Q$  to  $K'$ . According to Lemma 33,  $(K', Q')$  is a tame field with

$$K'Q' = LQ \quad \text{and} \quad v_{Q'}K' = v_Q L. \quad (7)$$

Hence  $(K', Q')$  is existentially closed in  $(L, Q)$  by Theorem 24.

Since the tame field  $K'$  is perfect, the algebraic function field  $K'.F|K'$  is separably generated. Therefore, we can write  $K'.F = K'(t_1, \dots, t_k, y)$ , where  $k = n - (d + r)$ , the elements  $t_1, \dots, t_k$  are algebraically independent over  $K'$ , and  $y$  is separable algebraic over  $K'(t_1, \dots, t_k)$ . Let  $f \in K'[t_1, \dots, t_k, Y]$  be the irreducible polynomial of  $y$  over  $K'[t_1, \dots, t_k]$ . For  $\underline{t} = (t_1, \dots, t_k)$ , we then have

$$f(\underline{t}, y) = 0 \quad \text{and} \quad \frac{\partial f}{\partial Y}(\underline{t}, y) \neq 0.$$

In view of (7), we can choose  $a'_1, \dots, a'_m \in K'$  such that

$$a'_i Q = a_i Q \quad \text{and} \quad v_Q a'_i = v_Q a_i \quad \text{for} \quad 1 \leq i \leq m.$$

We write the elements  $a_i$  as follows:

$$a_i = \frac{g_i(\underline{t}, y)}{h_i(\underline{t})} \quad \text{for } 1 \leq i \leq m,$$

where  $g_i$  and  $h_i$  are polynomials over  $K'$ , with  $h_i(\underline{t}) \neq 0$ . Since  $(K', Q')$  is existentially closed in  $(L, Q)$ , there exist elements

$$t'_1, \dots, t'_k, y' \in K'$$

such that

- (i)  $f(t', y') = 0$  and  $\frac{\partial f}{\partial Y}(t', y') \neq 0$ ,
- (ii)  $h_i(t') \neq 0$  for  $1 \leq i \leq m$ ,
- (iii)  $\frac{g_i(t', y')}{h_i(t')}Q = a'_iQ$  and  $v_Q \frac{g_i(t', y')}{h_i(t')} = v_Q a'_i$  for  $1 \leq i \leq m$ ,

since these assertions are true in  $L$  for  $\underline{t}, y$  in the place of  $\underline{t}', y'$ .

Now let  $K_1$  be the subfield of  $K'$  which is generated over  $K$  by the following elements:

- $x_1, \dots, x_r, y_1, \dots, y_d$ ,
- $a'_1, \dots, a'_m, t'_1, \dots, t'_k, y'$ ,
- the coefficients of  $f$ ,  $g_i$  and  $h_i$  for  $1 \leq i \leq m$ .

Let  $P_1$  denote the restriction of  $Q$  to  $K_1$ . We note that  $K_1$  is a finite extension of  $K_0$ . Hence according to Corollary 26,  $v_{P_1}K_1$  is a subgroup of  $v_Q L$ , finitely generated over  $v_\varphi K$  with  $v_{P_1}K_1/v_\varphi K$  of rational rank  $r$ . Similarly,  $K_1P_1$  is a subfield of  $LQ$  and finitely generated of transcendence degree  $d$  over  $K_\varnothing$ .

At this point we may forget about the field  $L$  and its place  $Q$ . Starting from  $(K_1, P_1)$  we will construct some henselian extension of  $(K_1, P_1)$  which will contain an isomorphic copy of  $F$ . The construction will be done in such a way that the restriction of the place to the embedded copy of  $F$  will satisfy the assertion of the theorem.

Choose  $d_1$  and  $r_1$  as in the assumption of Theorem 1. We take  $d_1 - d$  elements  $y_{d+1}, \dots, y_{d_1}$ , algebraically independent over  $K_1$ , and set  $K_2 := K_1(y_{d+1}, \dots, y_{d_1})$ . We extend  $P_1$  to a place  $P_2$  on  $K_2$  such that the value group does not change and the residue field  $K_2P_2$  becomes a purely transcendental extension of  $K_1P_1$  of transcendence degree  $d_1 - d$ . This can be done by an application of Lemma 25.

Next we adjoin  $r_1 - r$  elements  $x_{r+1}, \dots, x_{r_1}$ , algebraically independent over  $K_2$ . We assume that an arbitrary ordering on  $\frac{1}{p^\infty}v_Q F \oplus \bigoplus_{r_1-r} \mathbb{Z}$  has been fixed. We take  $\alpha_1, \dots, \alpha_{r_1-r}$  to be generators of that group over  $v_{P_2}K_2 = v_Q K_1$ ; then  $\alpha_1, \dots, \alpha_{r_1-r}$  are rationally independent over  $\frac{1}{p^\infty}v_Q F$ . By Lemma 25, there is an extension  $P_3$  of  $P_2$  to  $K_3 := K_2(x_{r+1}, \dots, x_{r_1})$  such that  $v_{P_3}x_{r+i} = \alpha_i$  for  $1 \leq i \leq r_1 - r$ , with  $v_{P_3}K_3 = v_{P_2}K \oplus \bigoplus_{r_1-r} \mathbb{Z} \subseteq \frac{1}{p^\infty}v_Q F \oplus \bigoplus_{r_1-r} \mathbb{Z}$  and  $K_3P_3 = K_2P_2$ .

By construction,  $(K_3, P_3)$  satisfies:

- $\text{trdeg } K_3|K = d + r + (r_1 - r) + (d_1 - d) = d_1 + r_1$ ,
- $K_3 P_3|K_\wp$  is finitely generated, with  $\text{trdeg } K_3 P_3|K_\wp = d + (d_1 - d) = d_1$ ,
- $v_{P_3} K_3 / v_\wp K$  is finitely generated, with  $\text{rr } v_{P_3} K_3 / v_\wp K = r + (r_1 - r) = r_1$ .

If  $d_1 + r_1 \geq 1$ , then  $\text{trdeg } K_3|K \geq 1$ , and Proposition 27 shows that  $(K_3, P_3)$  admits an immediate extension of transcendence degree  $n - (d_1 + r_1)$ . If  $d_1 + r_1 = 0$ , then our construction yields  $K_3 = K_1$  which is a finite extension of  $K_0 = K$ . In this case, the existence of such an immediate extension is guaranteed by the additional assumption at the end of our theorem. Now we pick any transcendence basis of this extension and take  $K_4$  to be the subextension which it generates over  $K_3$ . Restricting the place to the so obtained field, we get an immediate extension  $(K_4, P_4)$  of  $(K_3, P_3)$ .

Now we take  $(K_5, P_5)$  to be the henselization of  $(K_4, P_4)$ . It remains to show that  $F$  can be embedded in  $K_5$  over  $K$ . Then  $P_5$  will induce a place  $P$  on  $F$  which satisfies the assertions of our theorem. In fact, we find an embedding of  $K_1.F$  over  $K_1$  in  $K_5$  as follows.

We choose elements  $t_1^*, \dots, t_k^* \in K_5$ , algebraically independent over  $K_1$ , so close to  $t'_1, \dots, t'_k$  that by the Implicit Function Theorem (which holds in every henselian field, cf. [P–Z], Theorem 7.4) we can find  $y^* \in K_5$  satisfying  $f(\underline{t}^*, y^*) = 0$  and being so close to  $y'$  that in addition, (ii) and (iii) hold for  $\underline{t}^*, y^*$  in the place of  $\underline{t}', y'$  and  $P_5$  in the place of  $Q$ . Since  $\underline{t}', y'$  satisfy (ii) and (iii), and these conditions define an open set in the valuation topology, such elements  $t_1^*, \dots, t_k^*, y^*$  can be found in  $K_5$ . The fact that  $t_1^*, \dots, t_k^*$  can even be chosen to be algebraically independent over  $K_1$  follows from the choice of the transcendence degree of  $K_5$  over  $K_1$  (which is  $(d_1 - d) + (r_1 - r) + n - (d_1 + r_1) = n - (d + r) = k$ ), and the fact that for any intermediate field  $K_1 \subset K'_1 \subsetneq K_5$  which is relatively algebraically closed in  $K_5$ , the elements of  $K_5 \setminus K'_1$  lie dense in  $K_5$ . Applying this fact inductively yields the result.

Now  $t_i \mapsto t_i^*$  ( $1 \leq i \leq k$ ) and  $y \mapsto y^*$  defines an embedding of  $K_1.F$  over  $K_1$  in  $K_5$ . We identify  $F$  with its image in  $K_5$  and take  $P$  to be the restriction of  $P_5$  to  $F$ . By construction,  $K_4(\underline{t}^*, y^*)$  is a finite algebraic extension of  $F$ , having the purely transcendental extension  $K_4(\underline{t}^*, y^*) P_5 = K_3 P_3$  of  $K_1 P_1$  of transcendence degree  $d_1 - d$  as its residue field, and the  $(r_1 - r)$ -extension  $v_{P_5} K_4(\underline{t}^*, y^*) = v_{P_3} K_3$  of  $v_{P_1} K_1$  as its value group. Further, it follows that  $[K_3 P_3 : FP]$  is finite and therefore,  $FP|K_\wp$  is finitely generated of transcendence degree  $d_1$ . It also follows that  $(v_{P_3} K_3 : v_P F)$  is finite and therefore,  $v_P F / v_\wp K$  is finitely generated of rational rank  $r_1$ . Thus,  $FP$  and  $v_P F$  satisfy conditions (a) and (b) of the theorem.

Finally, we have to check the conditions on the elements  $a_i$ . After identifying  $K_1.F$  with its image in  $K_5$ , we have that

$$a_i = \frac{g_i(\underline{t}^*, y^*)}{h_i(\underline{t}^*)}.$$

Now the result follows from assertion (iii) (with  $P_5$  replaced by  $Q$ ) for  $\underline{t}^*, y^*$ , together with  $a'_i P_5 = a'_i Q = a_i Q$  and  $v_{P_5} a'_i = v_Q a'_i = v_Q a_i$  ( $1 \leq i \leq m$ ).  $\square$

### 3.2 Proof of Theorem 3

For the **proof of Theorem 3**, we modify the above proof in the following way. We replace  $(L, Q)$  by a maximal algebraic extension still having  $F^{1/p^\infty} Q = (FQ)^{1/p^\infty}$  as its residue field. Such an extension will have a divisible value group (cf. Section 2.3 of [K5]). The new  $(L, Q)$  is again a tame field, by Lemma 30.

Let us first assume that  $\wp$  is trivial. Then we take  $K'$  to be the relative algebraic closure of  $K(x_1, y_1, \dots, y_d)$  in  $L$ , and  $Q'$  the restriction of  $Q$ . By Lemma 33,  $(K', Q')$  is a tame field with  $K'Q' = LQ = (FQ)^{1/p^\infty}$  and  $v_{Q'} K' = \mathbb{Q} v x_1$ . We choose  $a'_1, \dots, a'_m \in K'$  such that

$$a'_i Q' = a_i Q \quad \text{for } 1 \leq i \leq m.$$

As  $v_{Q'} K'$  may be smaller than  $v_Q L$ , it may not be possible to choose the  $a'_i$  such that also  $v_{Q'} a'_i = v_Q a_i$ . Since a divisible ordered abelian group is existentially closed in every ordered abelian group extension, we can again apply Theorem 24. But we have to replace (iii) by

$$(iii) \quad \frac{g_i(\underline{t}', y')}{h_i(\underline{t}')} Q = a'_i Q \quad \text{for } 1 \leq i \leq m.$$

Therefore, we cannot preserve information about the values  $v_Q a_i$ . On the other hand, we gain the freedom to extend the value group of  $(K_1, P_1)$  (which is a finite extension of  $K(x_1, y_1, \dots, y_d)$  and thus has value group isomorphic to  $\mathbb{Z}$ ) by  $r_1 - 1$  new copies of  $\mathbb{Z}$ , where  $r_1$  can be chosen freely between 1 and  $n - d_1$ .

In the case of  $\wp$  being non-trivial, we take  $K'$  to be the relative algebraic closure of  $K(y_1, \dots, y_d)$  in  $L$  and proceed as above. In this case, the value group of  $(K_1, P_1)$  is a finite extension of  $v_\wp K$ . In the subsequent construction, we extend it by  $r_1$  new copies of  $\mathbb{Z}$ , where  $r_1$  can be chosen freely between 0 and  $n - d_1$ .  $\square$

### 3.3 Proof of Theorem 7

In view of Theorem 1, we only have to show that there is a zero-dimensional place  $P$  which satisfies (4) and for which  $v_P F$  is equal to the group  $\Gamma$  which is described in assertion (b) of Theorem 7, and is finitely generated over  $v_\wp K$ . First, we use Theorem 1 to find a place  $Q_1$  such that

- $v_{Q_1} a_i = v_Q a_i$  for  $1 \leq i \leq m$ ,
- $v_{Q_1} F$  is a subgroup of the  $p$ -divisible hull of  $v_Q F$ , finitely generated over  $v_\wp K$ ,
- $FQ_1 | K_\wp$  is finitely generated.

Now we have to construct a zero-dimensional place from  $Q_1$ . The idea is to find a zero-dimensional place on the function field  $FQ_1|K_\wp$  and then to compose it with  $Q_1$ .

We take  $K'$  to be the algebraic closure of  $K_\wp$  in the algebraic closure of  $FQ_1$ . Since  $K'$  is algebraically closed, it is existentially closed in  $F' := K'.(FQ_1)$ . Hence, we can apply Theorem 23 to the function field  $F'|K'$ . This gives us a  $K'$ -rational place  $Q'_2$  of  $K'.(FQ_1)$ . Its restriction to  $FQ_1$  is a zero-dimensional place of  $FQ_1|K_\wp$ . We use Theorem 3 to change it to a discrete zero-dimensional place  $Q_2$  of  $FQ_1|K_\wp$ . Now  $Q_1Q_2$  is indeed a zero-dimensional place of  $F$ . Its value group  $v_{Q_1Q_2}F$  contains  $v_{Q_2}FQ_1 = \mathbb{Z}$  as a convex subgroup, and  $v_{Q_1Q_2}F/v_{Q_2}FQ_1$  is isomorphic to  $v_{Q_1}F$ . We set  $P = Q_1Q_2$ , so  $\Gamma := v_P F = v_{Q_1Q_2}F$  is as described in assertion (b) of Theorem 7. Since  $Q_2$  is trivial on  $K_\wp$ , we have that  $v_{Q_1Q_2}K = v_\wp K$  and that  $v_P F/v_\wp K = v_{Q_1Q_2}F/v_{Q_1Q_2}K$  still has convex subgroup  $v_{Q_2}FQ_1 = \mathbb{Z}$  such that the quotient is  $v_{Q_1}F$ . Hence, also  $v_P F/v_\wp K$  is finitely generated.

However, we have to be more careful in order to satisfy the condition on the values. We choose a  $\mathbb{Z}$ -basis  $\gamma_1, \dots, \gamma_\ell$  of the group generated by the values  $v_Q a_1, \dots, v_Q a_m$ , and elements  $b_1, \dots, b_\ell \in F$  such that  $v_{Q_1} b_i = \gamma_i$ . Since these values are rationally independent and since  $v_{Q_1} F$  is a quotient of  $v_{Q_1Q_2} F$  by a convex subgroup, it follows that sending  $v_{Q_1} b_i$  to  $v_{Q_1Q_2} b_i$  induces an order preserving embedding  $\iota$  of  $\Gamma$  in  $v_{Q_1Q_2} F$ . We have to choose  $Q_2$  in such a way that  $\iota v_{Q_1} a_i = v_{Q_1Q_2} a_i$  for  $1 \leq i \leq m$ . By our choice of the  $\gamma_i$  there are integers  $e_{i,j}$  such that

$$v_{Q_1} a_i = \sum_{j=1}^{\ell} e_{i,j} \gamma_j = v \prod_{j=1}^{\ell} b_j^{e_{i,j}},$$

whence

$$v_{Q_1} a'_i = 0 \quad \text{for} \quad a'_i := a_i^{-1} \prod_{j=1}^{\ell} b_j^{e_{i,j}}.$$

That is,  $a'_i Q_1 \neq 0$ , and by Theorem 23, we can choose  $Q_2$  such that  $a'_i Q_1 Q_2 \neq 0$  for  $1 \leq i \leq m$ . That gives us  $v_{Q_1Q_2} a'_i = 0$  and consequently,

$$v_{Q_1Q_2} a_i = v_{Q_1Q_2} \prod_{j=1}^{\ell} b_j^{e_{i,j}} = \sum_{j=1}^{\ell} e_{i,j} v_{Q_1Q_2} b_j = \sum_{j=1}^{\ell} e_{i,j} \iota v_{Q_1} b_j = \iota \sum_{j=1}^{\ell} e_{i,j} v_{Q_1} b_j = \iota v_{Q_1} a_i.$$

This completes the proof of Theorem 7.

### 3.4 Proof of Theorem 13

The proof is an adaptation of the last part of the proof of Theorem 1. We take a function field  $F|K$  with a rational place  $Q$  which admits smooth local uniformization. Further, we take elements  $a_1, \dots, a_m \in \mathcal{O}_Q$ . Then there is an affine model of  $F$  with coordinate ring  $K[x_1, \dots, x_k]$  such that  $x_1, \dots, x_k \in \mathcal{O}_Q$ , the point  $(x_1 Q, \dots, x_k Q)$  is smooth and  $K$ -rational, and  $a_1, \dots, a_m \in K[x_1, \dots, x_k]$ . Among the elements  $x_i$  we can choose a

transcendence basis  $t_1, \dots, t_n$  and can rewrite the original polynomial relations as polynomial relations with polynomials  $f_1, \dots, f_\ell \in K[t_1, \dots, t_n][Y_1, \dots, Y_\ell]$ , satisfied by the remaining  $x_i$ 's, which we now call  $y_1, \dots, y_\ell$ . These elements satisfy the hypothesis of the multidimensional Hensel's Lemma, namely,

$$\left( \det \left( \frac{\partial f_i}{\partial Y_j}(y_1, \dots, y_\ell) \right)_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq \ell}} \right) Q = \det \left( \frac{\partial(f_i Q)}{\partial Y_j}(y_1 Q, \dots, y_\ell Q) \right)_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq \ell}} \neq 0$$

(cf. Sections 3 and 4 of [K4]). Here,  $f_i Q$  denotes the polynomial whose coefficients are obtained from the corresponding coefficients of  $f_i$  by an application of  $Q$ .

Now we take an arbitrary place  $P'$  on a rational function field  $K(z_1, \dots, z_n)$  with residue field  $K$  and such that  $z_1, \dots, z_n \in \mathcal{M}_{P'}$ . In fact, we can choose  $P'$  to be discrete (since  $K((z_1))$  is of infinite transcendence degree over  $K(z_1)$ ), or of maximal rank, or with any finitely generated value group of rational rank  $n$ . (For all possible choices, see [K5].) We take  $(F', P')$  to be the henselization of  $(K(z_1, \dots, z_n), P')$ .

The elements  $t_1 Q + z_1, \dots, t_n Q + z_n$  are algebraically independent over  $K$ , so  $t_i \mapsto t_i Q + z_i$  induces an isomorphism from  $K(t_1, \dots, t_n)$  onto  $K(z_1, \dots, z_n)$ . This sends the polynomials  $f_i$  to polynomials  $f_i^*$  with coefficients in  $K[z_1, \dots, z_n]$ , and these coefficients have the same images under  $P'$  as the original coefficients have under  $Q$ . Hence the new polynomials  $f_i^*$  also satisfy the hypothesis of the multidimensional Hensel's Lemma:

$$\det \left( \frac{\partial(f_i^* P')}{\partial Y_j}(y_1 Q, \dots, y_\ell Q) \right)_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq \ell}} \neq 0$$

with  $(y_1 Q, \dots, y_\ell Q) \in K^\ell \subseteq (F' P')^\ell$ . Therefore, by the multidimensional Hensel's Lemma (which holds in every henselian field) there is a common zero  $(y'_1, \dots, y'_\ell) \in (F')^\ell$  of the  $f_i^*$  such that  $y'_i P' = y_i Q$ ,  $1 \leq i \leq \ell$ . So the above constructed isomorphism can be extended to an embedding of  $F$  in  $F'$ . We identify  $F$  with its image and take  $P$  to be the restriction of  $P'$  to  $F$ . It is discrete if  $P'$  is, and of maximal rank if  $P'$  is. Since  $t_i P = (t_i Q + z_i) P' = t_i Q$  and  $y_i P = y'_i P' = y_i Q$ , and since  $a_j \in K[t_1, \dots, t_n, y_1, \dots, y_\ell]$ , we also find that  $a_j P = a_j Q$  for  $1 \leq j \leq m$ . This proves our theorem.  $\square$

### 3.5 Proofs for Sections 1.3 and 1.4

We start with the

**Proof of Theorem 23:** We adapt the proof of the lemma on p. 190 of [K-P]. Assume that  $K$  is existentially closed in  $F$ . It is well known and easy to prove that this implies that  $F|K$  is separable. Pick a separating transcendence basis  $x_1, \dots, x_d$  and  $y \in F$  separable algebraic over  $K(x_1, \dots, x_d)$  such that  $F = K(x_1, \dots, x_d, y)$ . Let  $f \in K[x_1, \dots, x_d, Y]$  be the irreducible polynomial of  $y$  over  $K[x_1, \dots, x_d]$ . We write

$$z_i = \frac{g_i(\underline{x}, y)}{h_i(\underline{x})} \quad \text{for } 1 \leq i \leq n,$$

where  $g_i$  and  $h_i$  are polynomials over  $K$ , with  $h_i(\underline{t}) \neq 0$ . Since  $x_1, \dots, x_d, y$  satisfy

$$f(\underline{x}, y) = 0 \quad \text{and} \quad \frac{\partial f}{\partial Y}(\underline{x}, y) \neq 0 \quad \text{and} \quad h_i(\underline{t}) \neq 0 \quad (1 \leq i \leq n)$$

in  $F$ , we infer from  $K$  being existentially closed in  $F$  that there are  $a_1, \dots, a_d, b$  in  $K$  such that

$$f(\underline{a}, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial Y}(\underline{a}, b) \neq 0 \quad \text{and} \quad h_i(\underline{a}) \neq 0 \quad (1 \leq i \leq n).$$

As in [K–P], we consider the field  $K((X_1)) \dots ((X_d))$  and embed  $F$  in  $L$  in such a way that  $x_i$  is sent to  $X_i + a_i$ . This induces a  $K$ -rational place  $P$  on  $F$  such that  $x_i P = a_i$ . It follows that  $z_i P \neq \infty$  ( $1 \leq i \leq n$ ).

Now suppose that we have already  $K$ -rational places  $P_1, \dots, P_k$  which are finite on  $z_1, \dots, z_n$ . Define  $z_{n+j} := (x_1 - x_1 P_j)^{-1} \in F$  for  $1 \leq j \leq k$ . By the above there exists a place  $P$  which is finite on  $z_1, \dots, z_{n+k}$ . It follows that  $x_1 P \neq x_1 P_j$  and hence  $P \neq P_j$  for  $1 \leq j \leq k$ . This shows that there are infinitely many  $K$ -rational places which are finite on  $z_1, \dots, z_n$ .  $\square$

For the case of one variable, we extend Theorem 23 as follows:

**Proposition 34** *Let  $F|K$  be a function field in one variable. Then  $K$  is existentially closed in  $F$  if and only if  $F$  admits infinitely many  $K$ -rational places.*

**Proof:** If  $K$  is existentially closed in  $F$ , then Theorem 23 shows that  $F$  admits infinitely many  $K$ -rational places.

For the converse, assume that  $F$  admits infinitely many  $K$ -rational places. Suppose that  $f_1, \dots, f_n \in K[X_1, \dots, X_m]$  have a common zero  $(a_1, \dots, a_m) \in F^m$ . If  $P$  is a  $K$ -rational place of  $F$  such that  $a_i P \neq \infty$  for  $1 \leq i \leq m$ , then  $(a_1 P, \dots, a_m P) \in K^m$  is a common zero of  $f_1, \dots, f_n$  since  $f_j(a_1 P, \dots, a_m P) = f_j(a_1, \dots, a_m) P = 0 P = 0$  for  $1 \leq j \leq n$ . Now it suffices to show that there are only finitely many  $K$ -rational places  $P$  of  $F$  for which  $a_i P = \infty$  for some  $i$ . But this is clear because  $a_i \mapsto \infty$  defines a unique place on  $K(a_i)$  (namely, the  $a_i^{-1}$ -adic place), and since  $F|K(a_i)$  is a finite extension (as  $a_i$  must be transcendental over  $K$ ), this place has only finitely many extensions to  $F$ .  $\square$

### Proof of Theorem 15:

1)  $\Leftrightarrow$  2): By the foregoing proposition, (LF'') is equivalent to 2).

2)  $\Rightarrow$  3): Since  $K(t)^h$  admits a  $K$ -rational place, so does every function field  $F|K$  which is contained in  $K(t)^h$ . Every such function field  $F|K$  is a function field in one variable. Hence by 2),  $K$  is existentially closed in  $F$ . It follows that  $K$  is existentially closed in  $K(t)^h$ .

3)  $\Rightarrow$  4): If  $K$  is existentially closed in  $K(t)^h$ , then  $K$  is existentially closed in  $K((t))$ , since this property is transitive and the following holds:

**Theorem 35** *The field  $K(t)^h$  is existentially closed in  $K((t))$ .*

This result follows from Theorem 2 in [ER]. Another proof will be given in [K2].

4)  $\Rightarrow$  5): Suppose that  $L$  is an extension field of  $K$  which admits a discrete  $K$ -rational place. Then it has a local parameter  $t$  and can be embedded over  $K(t)$  in  $K((t))$ . Hence if  $K$  is existentially closed in  $K((t))$ , then it is also existentially closed in  $L$ .

5)  $\Rightarrow$  2) is trivial.  $\square$

**Proof of Proposition 16:** Let  $v$  be the henselian valuation on  $K$ . Take any  $|K|^+$ -saturated elementary extension  $(K^*, v^*)$  of  $(K, v)$ , where  $|K|^+$  denotes the successor cardinal of the cardinality of  $K$ . Then  $(K^*, v^*)$  is henselian. Moreover,  $v^*K^*$  will contain an element  $\alpha$  which is bigger than every element in  $vK$ .

Now we consider  $K(t)^h$  with the valuation  $w := v_t \circ v$  which is the composition of the  $t$ -adic valuation  $v_t$  and the valuation  $v$  on its residue field  $K = K(t)^h P_{v_t}$ . Since  $v$  is henselian,  $K(t)^h$  is also the henselization of  $K(t)$  with respect to  $w$ .

The value  $wt$  is bigger than every element in  $wK = vK$ . Hence, sending  $wt$  to  $\alpha$  induces an order preserving isomorphism from  $wK(t) = vK \oplus \mathbb{Z}wt$  onto  $vK \oplus \mathbb{Z}\alpha$ . Consequently, sending  $t$  to some element  $t^* \in K^*$  with  $v^*t^* = \alpha$  induces a valuation preserving embedding of  $K(t)$  in  $K^*$  (apply Lemma 25). By the universal property of henselizations, this embedding extends to an embedding of  $K(t)^h$  in the henselian field  $K^*$ .

Since all existential sentences are preserved by this embedding, and since  $K$  is existentially closed in  $K^*$ , it now follows that  $K$  is existentially closed in  $K(t)^h$ .  $\square$

**Proof of Theorem 17:** The implication 3)  $\Rightarrow$  2) is trivial, and 2)  $\Rightarrow$  1) follows from Theorem 15 since  $K((t)) = k((\mathbb{Z}))$ . Hence it remains to show the implication 1)  $\Rightarrow$  3).

We take any field extension  $L$  of  $K$  which admits a  $K$ -rational place  $P$ . We extend  $P$  to the perfect hull  $L^{1/p^\infty}$  of  $L$ . Since  $L^{1/p^\infty}P = (LP)^{1/p^\infty} = K^{1/p^\infty}$  and  $K$  is perfect by assumption, we find that  $L^{1/p^\infty}P = K$ . Now we take  $(L_1, P)$  to be a maximal immediate algebraic extension of  $(L^{1/p^\infty}, P)$ . Hence, we still have  $L_1P = K$ . By Corollary 31,  $(L_1, P)$  is a tame field. By adjoining suitable  $n$ -th roots of elements in  $L_1$ , we can further extend to a valued field  $(L_2, P)$  such that  $v_PL_2$  is divisible and  $L_2P = K$ . By Lemma 29, also  $(L_2, P)$  is a tame field.

Now take any  $x \in L_2$  which is transcendental over  $K$  and let  $L_0$  be the relative algebraic closure of  $K(x)$  in  $L_2$ . Since  $L_2P = K$ ,  $P$  must be non-trivial on  $L_0$ . We have  $L_0P = K$  and by Lemma 33,  $(L_0, v_P)$  is a tame field and  $v_PL_0$  is divisible. Consequently,  $v_PL_0$  is existentially closed in  $v_PL_2$ . Trivially,  $L_0P = K$  is existentially closed in  $L_2P = K$ . So we can employ Theorem 24 to deduce that  $L_0$  is existentially closed in  $L_2$ .

Now it suffices to prove that  $K$  is existentially closed in  $L_0$  because then by transitivity,  $K$  is existentially closed in  $L_2$  and thus also in its subfield  $L$ . We only have to show that

$K$  is existentially closed in every subfield  $F$  of  $L_0$  which is finitely generated over  $K$ . Since  $L_0$  is an algebraic extension of  $K(x)$  and  $P$  is trivial on  $K$ , it follows that  $P$  is discrete on  $F$ . Since  $K$  is large, we can now infer from Theorem 15 that  $K$  is existentially closed in  $F$ . This completes our proof.  $\square$

## 4 Appendix

Let  $\mathcal{L}_0$  be a first order language, and  $\mathcal{L}$  an extension of  $\mathcal{L}_0$  by new relation symbols. Take  $M_0$  to be any  $\mathcal{L}_0$ -structure. Further, let  $\mathcal{T}$  be a set of universal  $\mathcal{L}$ -sentences, and  $\mathcal{A}$  a set of existential  $\mathcal{L}$ -sentences.

Now let  $X$  be the set of  $\mathcal{L}$ -expansions  $M$  of  $M_0$  such that  $M \models \mathcal{T}$ . On  $X$ , we consider the topology  $\mathcal{X}_{\mathcal{A}}$  whose basic open sets are the sets of the form

$$\{M \in X \mid M \models \mathcal{A}'\}, \quad \mathcal{A}' \text{ a finite subset of } \mathcal{A}.$$

Replacing  $\mathcal{A}$  by its closure under finite conjunctions if necessary, we may assume that all basic open sets are of the form

$$X_{\varphi} := \{M \in X \mid M \models \varphi\},$$

where  $\varphi \in \mathcal{A}$ .

**Theorem 36**  *$(X, \mathcal{X}_{\mathcal{A}})$  is quasi-compact.*

Proof: Take a collection  $\{X_{\varphi_i} \mid i \in I\}$  of basic open sets, with  $\varphi_i \in \mathcal{A}$ . Assume that

$$X = \bigcup_{i \in J} X_{\varphi_i}$$

does not hold for any finite subset  $J$  of  $I$ . Then we have to show that it also does not hold for  $J = I$ .

By our assumption, for every finite subset  $J$  of  $I$  there is an expansion  $M_J \in X$  which is not in  $\bigcup_{i \in J} X_{\varphi_i}$ . That is,  $M_J \models \bigwedge_{i \in J} \neg \varphi_i$ . We take  $\mathcal{T}_0$  to be the elementary  $\mathcal{L}_0(M_0)$ -theory of  $M_0$ , where  $\mathcal{L}_0(M_0)$  is the language obtained from  $\mathcal{L}_0$  by adjoining a constant symbol for every element of  $M_0$ . We see that for every finite subset  $J$  of  $I$ , the theory  $\mathcal{T}_0 \cup \mathcal{T} \cup \{\neg \varphi_i \mid i \in J\}$  has a model, namely,  $M_J$ . By the semantical compactness theorem of first order logic, we conclude that also the theory  $\mathcal{T}_0 \cup \mathcal{T} \cup \{\neg \varphi_i \mid i \in I\}$  has a model  $M^*$ .

Since  $M^* \models \mathcal{T}_0$ , we know that  $M_0$  is an elementary substructure of the  $\mathcal{L}_0$ -reduct of  $M^*$ . Let us denote by  $M'$  the  $\mathcal{L}$ -structure which we obtain by restricting the new relations of  $M^*$  to the universe of  $M_0$ . Then  $M_0$  is the  $\mathcal{L}_0$ -reduct of  $M'$ , that is,  $M'$  is an  $\mathcal{L}$ -expansion of  $M_0$ . Since  $\mathcal{T}$  consists of universal sentences, we also have that  $M' \models \mathcal{T}$ .

Hence,  $M' \in X$ . But for all  $i \in I$ ,  $M^* \models \neg\varphi_i$  and since  $\neg\varphi_i$  is a universal sentence, we also have that  $M' \models \neg\varphi_i$ . Therefore,

$$M' \notin \bigcup_{i \in I} X_{\varphi_i}.$$

□

Replacing  $\mathcal{T}$  by  $\mathcal{T} \cup \{\varphi\}$  for any quantifier free sentence  $\varphi \in \mathcal{A}$ , we obtain:

**Corollary 37** *If  $\varphi \in \mathcal{A}$  is quantifier free, then  $X_\varphi$  is quasi-compact. Hence if  $\mathcal{A}$  consists only of quantifier free sentences, then every basic open set is quasi-compact.*

Now we consider the following conditions on  $\mathcal{A}$ :

- (T<sub>0</sub>) for all  $M_1, M_2 \in X$ ,  $M_1 \neq M_2$ , there is some  $\varphi \in \mathcal{A}$  such that  $M_1 \notin X_\varphi \ni M_2$  or  $M_2 \notin X_\varphi \ni M_1$ .
- (NEG)  $\mathcal{A}$  is closed under negation.

Note: if  $\mathcal{A}$  satisfies (NEG), then it consists solely of quantifier free sentences.

**Proposition 38** *If  $\mathcal{A}$  satisfies (T<sub>0</sub>), then  $(X, \mathcal{X}_\mathcal{A})$  is a T<sub>0</sub>-space. If in addition,  $\mathcal{A}$  satisfies (NEG), then  $(X, \mathcal{X}_\mathcal{A})$  is a totally disconnected compact Hausdorff space and its basic open sets are both open and closed.*

Proof: The first assertion is obvious. Now assume that both (T<sub>0</sub>) and (NEG) hold. Since  $\mathcal{A}$  is closed under negation, the complement  $X_{\neg\varphi}$  of a basic open set  $X_\varphi$  is also basic open. Therefore, each set  $X_\varphi$  is open and closed, and  $(X, \mathcal{X}_\mathcal{A})$  is totally disconnected. If  $M_1, M_2 \in X$  such that  $M_1 \neq M_2$ , then by (T<sub>0</sub>), there is some  $\varphi \in \mathcal{A}$  such that  $M_1 \in X_\varphi$  and  $M_2 \in X_{\neg\varphi}$ . This shows that  $(X, \mathcal{X}_\mathcal{A})$  is Hausdorff. The compactness follows from Theorem 36. □

Let us observe the following fact, which we will not need any further:

**Corollary 39** *If  $\mathcal{A}$  satisfies (T<sub>0</sub>) and (NEG), then  $\mathcal{X}_\mathcal{A} = \mathcal{X}_\mathcal{Q}$  where  $\mathcal{Q}$  denotes the set of all quantifier free elementary  $\mathcal{L}$ -sentences.*

Proof: It is clear that  $\mathcal{Q}$  satisfies (NEG). Since  $\mathcal{A} \subseteq \mathcal{Q}$ , we know that  $\mathcal{Q}$  also satisfies (T<sub>0</sub>). Hence,  $(X, \mathcal{X}_\mathcal{Q})$  is a totally disconnected compact Hausdorff space. Since the same holds for  $(X, \mathcal{X}_\mathcal{A})$  and since  $\mathcal{X}_\mathcal{A} \subseteq \mathcal{X}_\mathcal{Q}$ , we must have that  $\mathcal{X}_\mathcal{A} = \mathcal{X}_\mathcal{Q}$ . □

**Theorem 40** *Suppose that  $\mathcal{A}$  satisfies (NEG), and that  $\mathcal{B}$  is a subset of  $\mathcal{A}$  which satisfies (T<sub>0</sub>). Then  $(X, \mathcal{X}_\mathcal{B})$  is a spectral space and  $\mathcal{X}_\mathcal{A}$  is its associated patch topology.*

Proof: If  $\mathcal{A}$  contains a subset  $\mathcal{B}$  which satisfies  $(T_0)$ , then also  $\mathcal{A}$  satisfies  $(T_0)$ . Thus, our assertion follows from Proposition 38 together with Proposition 7 of [H].  $\square$

In order to apply this result to spaces of valuation rings on a fixed field  $F$ , we take  $\mathcal{L}_0$  to be the language of rings together with a set  $\mathcal{F}$  of constant symbols for all elements of  $F$ , i.e.,  $\mathcal{L}_0 = \{+, -, \cdot, 0, 1\} \cup \mathcal{F}$ . Further, we set  $\mathcal{L} = \mathcal{L}_0 \cup \{\mathcal{O}\}$ , where  $\mathcal{O}$  is a unary predicate symbol. We take  $\mathcal{T}_v$  to consist of the following universal  $\mathcal{L}$ -sentences which say that (the interpretation of)  $\mathcal{O}$  is a subring which is a valuation ring:

- 1)  $\forall x \forall y : (\mathcal{O}(x) \wedge \mathcal{O}(y)) \rightarrow (\mathcal{O}(x - y) \wedge \mathcal{O}(xy)),$
- 2)  $\forall x \forall y : xy = 1 \rightarrow (\mathcal{O}(x) \vee \mathcal{O}(y)).$

Further, we let  $\mathcal{T}_S$  be an arbitrary set of universal  $\mathcal{L}$ -sentences; they single out a universally definable subset  $S$  of valuation rings on  $F$ . We set  $M_0 = F$ ,  $\mathcal{T} = \mathcal{T}_v \cup \mathcal{T}_S$ ,

$$\mathcal{A} = \{\mathcal{O}(a), \neg \mathcal{O}(a) \mid a \in \mathcal{F}\} \quad \text{and} \quad \mathcal{B} = \{\mathcal{O}(a) \mid a \in \mathcal{F}\}.$$

In this setting, every expansion  $M \in X$  is given by the choice of a valuation ring  $\mathcal{O}$  of  $F$  which satisfies  $\mathcal{T}_S$ . Hence we have a bijection between  $X$  and the set  $S(F; \mathcal{T}_S)$  of all valuation rings on  $F$  which satisfy  $\mathcal{T}_S$ , and we identify these sets. Then  $\mathcal{X}_{\mathcal{B}}$  is the Zariski topology on  $S(F; \mathcal{T}_S)$ . We note that for  $a \neq 0$ ,  $a \notin \mathcal{O}$  is equivalent to  $a^{-1} \in \mathcal{O}$ . Therefore,  $\mathcal{X}_{\mathcal{A}}$  is the patch topology on  $S(F; \mathcal{T}_S)$ , as defined by the basic open sets (2) in the introduction.

Suppose that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are two distinct valuation rings on  $F$ . Then there is some  $a \in F$  such that  $a \in \mathcal{O}_1 \setminus \mathcal{O}_2$  or  $a \in \mathcal{O}_2 \setminus \mathcal{O}_1$ . So we see that  $\mathcal{B}$  satisfies  $(T_0)$ . Clearly,  $\mathcal{A}$  satisfies (NEG). Hence by Theorem 40, we obtain:

**Theorem 41** *The Zariski space  $S(F; \mathcal{T}_S)$  together with the Zariski topology given by the basic open sets (1) is a spectral space, and the topology given by the basic open sets (2) is its associated patch topology.*

Taking  $\mathcal{T}_S = \emptyset$ , we see that these assertions hold in particular for the Zariski space of all valuation rings on a fixed field  $F$ .

As we have constant symbols for all elements of  $K$  in our language  $\mathcal{L}$ , we can take  $\mathcal{T}_S$  to be a set of quantifier free  $\mathcal{L}$ -sentences in  $\mathcal{T}$  which state that  $\mathcal{O} \cap K$  is the valuation ring  $\mathcal{O}_{\wp}$  of a given place  $\wp$  on  $K$ . Then  $S(F; \mathcal{T}_S) = S(F|K; \wp)$ .

**Corollary 42** *The assertions of Theorem 41 hold in particular for  $S(F|K; \wp)$  and for  $S(F|K)$ .*

Let us demonstrate the usefulness of Theorem 40 by two more applications, the first of which can be seen as a generalization of Theorem 41. In both applications, let  $R$  be a commutative ring with unity.

The set  $\text{Spv}(R)$  of all valuations (in the sense of [HU–KN]) on  $R$  is called the **valuation spectrum** of  $R$ . As  $R$  will in general not be a field, instead of a unary predicate for the valuation ring we have to use a binary relation for what is called **valuation divisibility**. We write  $x|y \iff vx \leq vy$ . In order to encode a valuation, the relation  $|$  has to satisfy the following universal axioms (cf. [HU–KN]):

- 1)  $\forall x \forall y : x|y \vee y|x$ ,
- 2)  $\forall x \forall y \forall z : (x|y \wedge y|z) \rightarrow x|z$ ,
- 3)  $\forall x \forall y \forall z : (x|y \wedge x|z) \rightarrow x|y + z$ ,
- 4)  $\forall x \forall y \forall z : x|y \rightarrow xz|yz$ ,
- 5)  $\forall x \forall y \forall z : (xz|yz \wedge 0 \nmid z) \rightarrow x|y$ ,
- 6)  $0 \nmid 1$ .

We take  $\mathcal{R}$  to be a set of constant symbols for all elements of  $R$ , and  $\mathcal{L}_0 = \{+, -, \cdot, 0, 1\} \cup \mathcal{R}$ . Further, we take  $\mathcal{L} = \mathcal{L}_0 \cup \{| \}$  with  $|$  a binary predicate symbol,  $M_0 = R$ , and  $\mathcal{T}$  to consist of the above axioms. One possible pair of topologies on the valuation spectrum is given by the sets

$$\mathcal{A} = \{a|b, a \nmid b \mid a, b \in \mathcal{R}\} \quad \text{and} \quad \mathcal{B} = \{a|b \mid a, b \in \mathcal{R}\}.$$

It is clear that  $\mathcal{B}$  satisfies  $(T_0)$ . Now Theorem 40 shows:

**Theorem 43** *The valuation spectrum of  $R$  is a spectral space, for the topology whose basic open sets are of the form  $\{v \in \text{Spv}(R) \mid va_1 \leq vb_1, \dots, va_k \leq vb_k\}$ . The basic open sets of its patch topology are of the form  $\{v \in \text{Spv}(R) \mid va_1 \leq vb_1, \dots, va_k \leq vb_k; vc_1 < vd_1, \dots, vc_\ell < vd_\ell\}$ .*

Note that Huber and Knebusch prefer to work with different topologies which essentially are obtained from the above by adding the condition  $a \neq 0$ . Yet the above theorem remains true (cf. [HU–KN]).

The **real spectrum** of the ring  $R$  is the set of preorderings  $P$  with support a prime ideal and satisfying  $P \cup -P = R$  and  $-1 \notin P$  (cf. [B–C–R], [C–R], [KN–S]). That is, it can be presented by all (interpretations of) unary predicates  $P$  which satisfy the following universal axioms:

- 1)  $\forall x \forall y : (P(x) \wedge P(y)) \rightarrow (P(x + y) \wedge P(xy))$ ,
- 2)  $\forall x : P(x^2)$ ,
- 3)  $\forall x : P(x) \vee P(-x)$ ,
- 4)  $\neg P(-1)$ ,
- 5)  $\forall x \forall y : (P(xy) \wedge P(-xy)) \rightarrow (P(x) \wedge P(-x)) \vee (P(y) \wedge P(-y))$ .

We take  $\mathcal{L}_0$  and  $M_0$  as before. Further, we take  $\mathcal{L} = \mathcal{L}_0 \cup \{P\}$  with  $P$  a unary predicate symbol, and  $\mathcal{T}$  to consist of the above axioms. The topologies on the real spectrum are given by the sets

$$\mathcal{A} = \{P(a), \neg P(a) \mid a \in \mathcal{R}\} \quad \text{and} \quad \mathcal{B} = \{P(a) \mid a \in \mathcal{R}\}.$$

Again,  $\mathcal{B}$  satisfies  $(T_0)$ . So Theorem 40 shows:

**Theorem 44** *The real spectrum of  $R$  is a spectral space, for the topology whose basic open sets are of the form  $\{P \in \operatorname{Sper}(R) \mid a_1, \dots, a_k \in P\}$ . The basic open sets of its patch topology are of the form  $\{P \in \operatorname{Sper}(R) \mid a_1, \dots, a_k \in P; b_1, \dots, b_\ell \notin P\}$ .*

A comparable approach to spectral spaces using model theory has been worked out by R. Berr in [BE].

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